Math 250 6.3 Volumes: ~~Volumes~~ The Disk/Washer Method
Volumes by Slicing

Objectives
1) Find volume of solid with known cross-sections.
2) Find volume of a solid of revolution using the disk method (cross-section is a circle)
3) Find volume of a solid of revolution using the washer method (cross section is a ring - empty center disk with empty center)

Primary concept: definite integral is accumulating the areas of cross sections which are perpendicular to the axis of integration in all disks, washers and cross-sections

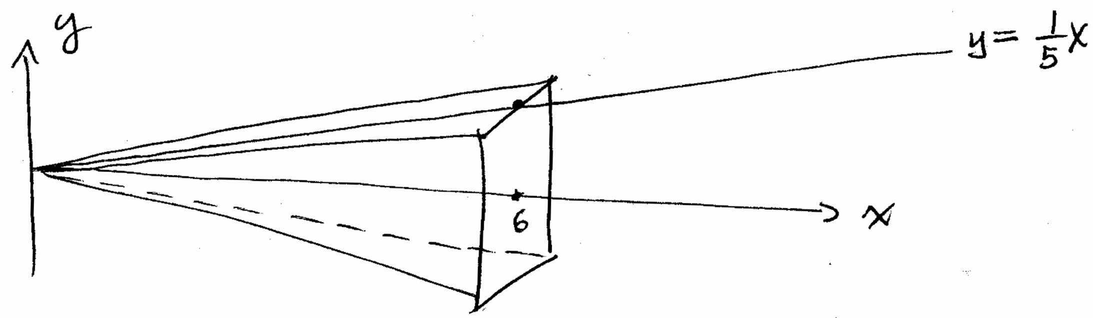
Notice:

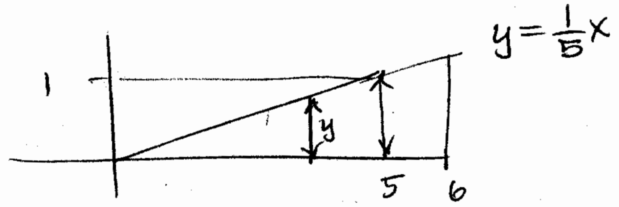
- Around what line do I revolve?
- Do I integrate in x or in y?
- Have I written my integrand using the variable with which I should integrate?
- Have I written my limits of integration with the correct variable?

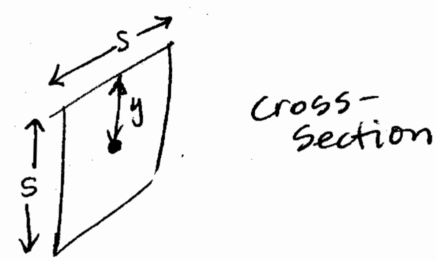
Math 250

(1) Calculate the volume of a pyramid with square cross-sections where the top side is given by $y = \frac{1}{5}x$ from $x=0$ to $x=6$.

yes







area of cross-section $= L \cdot W = s^2$ because square

where $s = 2y$

area of cross-section $= s^2 = (2y)^2 = 4y^2 = 4\left(\frac{1}{5}x\right)^2 = \frac{4}{25}x^2$

accumulate cross-sections from $x=0$ to $x=6$.

$$\int_{x=0}^{x=6} [s(x)]^2 \, dx$$

$$= \int_0^6 \frac{4}{25}x^2 \, dx = \frac{4}{25} \cdot \frac{1}{3}x^3 \Big|_0^6 = \frac{4}{75}(216 - 0) = \boxed{\frac{288}{25}}$$

Geometry Formula:

$\left(\text{Area of base}\right) \times (\text{height}) \times \left(\frac{1}{3}\right) \quad \left(\frac{4}{25} \cdot 6^2\right)(6)\left(\frac{1}{3}\right) = \frac{288}{25}$

M250

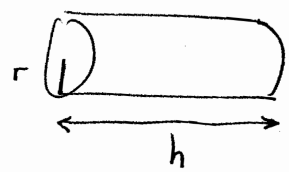
Background:

Volume of a cylinder

$V = (\text{area of base}) \cdot \text{height}$

$V = \pi r^2 h$

Same if sideways, only "base" is on the left end (or right end, or any cross-section)



$V = \pi r^2 h$

(2) Example: cylinder has $r=3$ and $h=4$

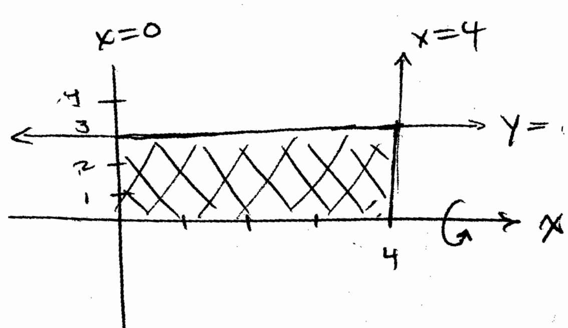
$V = \pi (3)^2 \cdot 4$

$\boxed{V = 36\pi \text{ unit}^3}$

Let's consider the cylinder as our most basic volume of revolution.

(3) Find the volume of the solid generated by revolving the region bounded by the given graphs about the x-axis.

$y=0$
$y=3$
$x=0$
$x=4$

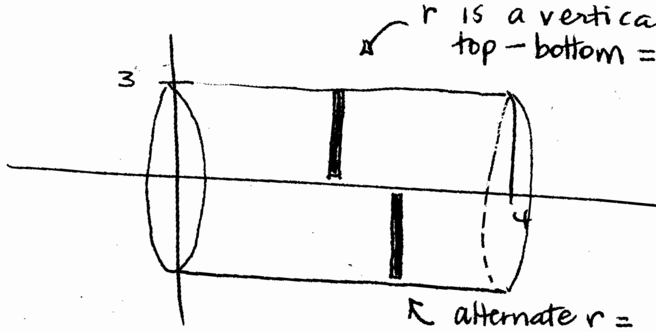


Step 1: sketch region

Step 2: rotate this region around x-axis.

(we get a cylinder)
$r=3$, $h=4$ $V=36\pi$

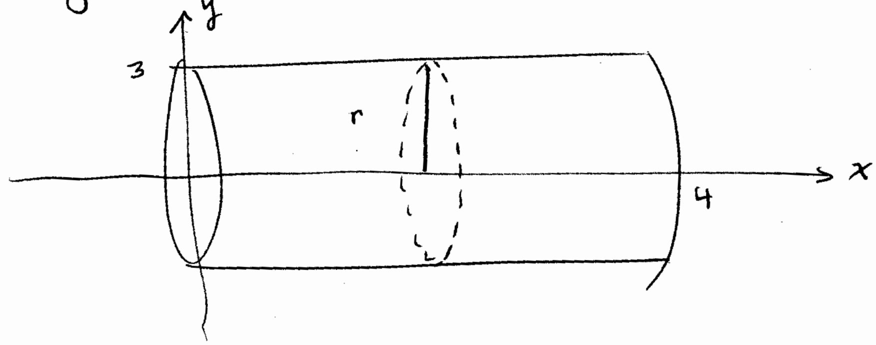
r is a vertical bar
top − bottom = $3-0=3$



alternate $r = 0-(-3)=3$

Notice that the radius is perpendicular to the axis of revolution.

Any cross section is a circle.
The radius of the circle is the height of the function, which in this case is always 3.



Step 3: We will use a definite integral to accumulate the areas of cross sections as x moves from 0 to 4.

$$\int_0^4 \pi r^2 \, dx$$

$$\int_0^4 \pi \cdot [y(x)]^2 \, dx$$

$$\int_0^4 \pi \cdot 3^2 \, dx$$

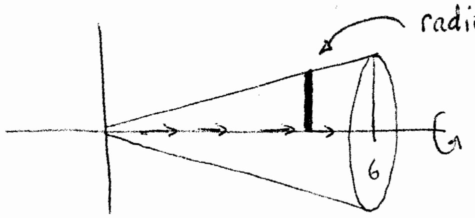
$$= 9\pi \int_0^4 dx$$

$$= 9\pi \left. x \right|_0^4$$

$$= 9\pi (4 - 0)$$

$$= \boxed{36\pi}$$

(4) Revolve $\begin{cases} y = \frac{1}{5}x \\ x = 6 \\ y = 0 \end{cases}$ around x-axis



radius is a vertical bar

$= \text{top } y - \text{bottom } y$

$= \frac{1}{5}x - 0$

$= \frac{1}{5}x$

accumulate in x-direction

cross-section is a circle

$A = \pi r^2$

radius = y-coord

$y = \frac{1}{5}x$

$A = \pi\left(\frac{1}{5}x\right)^2$

$$\text{volume} \int_{x=0}^{x=6} \underbrace{\left(\text{area of cross-section}\right)}_{\text{as an expression using } x} dx$$

$$= \int_{x=0}^{6} \pi\left(\frac{1}{5}x\right)^2 dx$$

$$= \frac{\pi}{25}\int_0^6 x^2\, dx$$

move constant multiple to front of integral

$$= \frac{\pi}{25}\cdot\frac{1}{3}x^3 \Big|_0^6$$

antidifferentiate

$$= \frac{\pi}{25}\cdot\frac{1}{3}\cdot(216 - 0)$$

evaluate using FTC

$$= \boxed{\frac{72\pi}{25}}$$

check by geometry: Volume of cone

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{6}{5}\right)^2 \cdot 6$

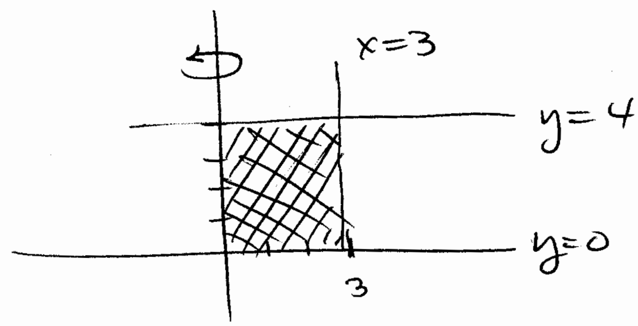
$V = \boxed{\frac{72\pi}{25}}$ ✓

M250

no ⑤ Find the volume of the solid generated by revolving the region bounded by the given graphs around the y-axis.

$x = 3$
$y = 0$
$y = 4$
$x = 0$

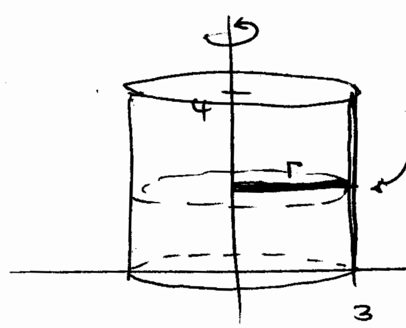
step 1: sketch region



step 2: revolve around y-axis

Notice radius is perpendicular to axis of revolution.

any cross-section is a circle.



radius is a horizontal bar
$= \text{right } x - \text{left } x$
$= 3 - 0$
$= 3$

The radius is always the x-coord 3.

step 3: accumulate in y

$$\int_0^4 \pi r^2 \, d\cancel{x}y$$

$$= \int_0^4 \pi 3^2 \, d\cancel{x}y$$

$$= \boxed{36\pi}$$

Math 250

But what if the radius is not constant?

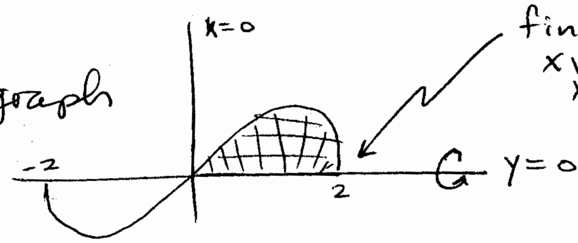
(6) $y = x\sqrt{4-x^2}$

$y = 0$

$x = 0.$

around x-axis

graph

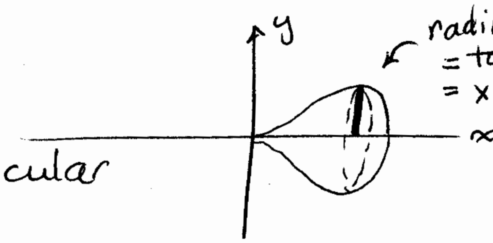


find x-ints

$x\sqrt{4-x^2} = 0$

$x = 0, -2, 2$

revolve:



radius is a vertical bar

$= \text{top } y - \text{bottom } y$

$= x\sqrt{4-x^2} - 0$

(turnip-shaped.)

radius is perpendicular to axis of rotation.

radius changes as x changes. $r(x) = x\sqrt{4-x^2}$

But: we can accumulate circular cross-sections using radius as a function of x.

$$\int_0^2 \pi r^2 \, dx$$ accumulate areas of circles

$$= \int_0^2 \pi \left(x\sqrt{4-x^2}\right)^2 dx$$ substitute $r(x)$

$$= \pi \int_0^2 x^2(4-x^2)\, dx$$ simplify square

$$= \pi \int_0^2 4x^2 - x^4 \, dx$$ dist

$$= \pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2$$ antidiff

$$= \pi\left[\left(\frac{32}{3} - \frac{32}{5}\right) - 0\right]$$ evaluate

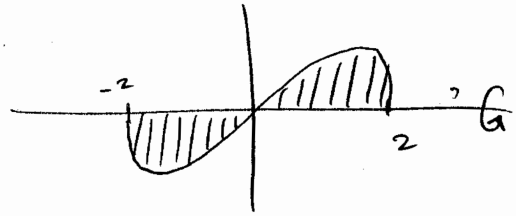
$$= \boxed{\frac{64\pi}{15}}$$

Math 250

No (7) $y = x\sqrt{4-x^2}$

$y = 0$

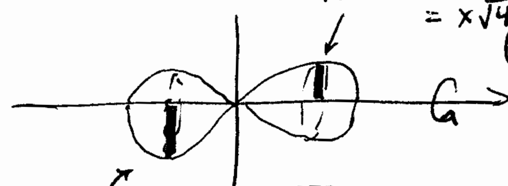
around x-axis



$$\int_{-2}^{2} \pi \left(x\sqrt{4-x^2}\right)^2 dx$$

radius = vertical bar

$= x\sqrt{4-x^2} - 0$



(hourglass, rounded top + bottom, or 3-D figure 8)

$r = 0 - x\sqrt{4-x^2}$

since radius is squared, negative disappears

$$= 2\int_{-2}^{2} \pi \left(x\sqrt{4-x^2}\right)^2 dx$$

because $r(x)$ is symmetric about the origin:

$-y = -x\sqrt{4-(-x)^2}$ $\quad (-x,-y)$

$-y = -x\sqrt{4-x^2}$ $\quad$ same as

$y = x\sqrt{4-x^2}$ $\quad (x,y)$

$$= \boxed{\frac{128\pi}{15}}$$

Math 250

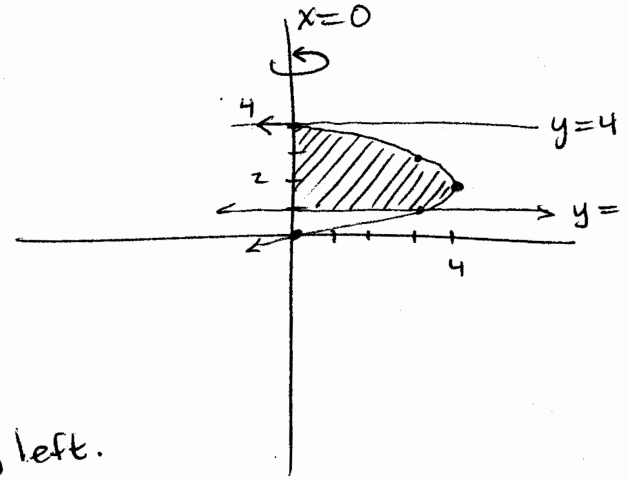
(9) $x = -y^2 + 4y$

$y = 1$

$y = 4$

$x = 0$

around y-axis



Graph $x = -y^2 + 4y$, a parabola opening left.

vertex $y = \frac{-b}{2a} = \frac{-4}{2(-1)} = \frac{-4}{-2} = 2$

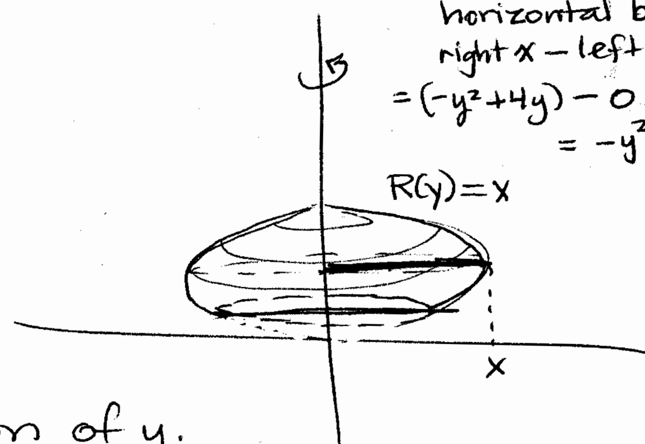
$$x = -(2)^2 + 4(2)$$
$$= -4 + 8$$
$$= 4$$

vertex $(4, 2)$

$a = -1 < 0$ opens left
standard shape

radius is a horizontal bar
right x − left x
$= (-y^2 + 4y) - 0$
$= -y^2 + 4y$

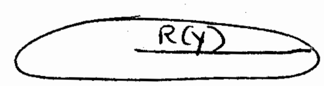
$R(y) = x$



radius is perpendicular to axis of revolution

radius $x = r(y)$ is a function of y.

$$\int_{y=1}^{y=4} \pi (r(y))^2 \, dy$$



accumulate areas of circles as y moves from $y=1$ to $y=4$.

$= \int_1^4 \pi (-y^2 + 4y)^2 \, dy$ subst radius

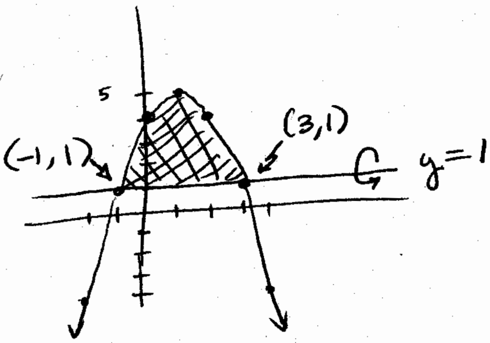
$= \pi \int_1^4 y^4 - 8y^3 + 16y^2 \, dy$ FOIL

$= \pi \left[\frac{1}{5}y^5 - 2y^4 + \frac{16}{3}y^3 \right] \Big|_1^4$ antidiff

$= \pi \left(\frac{512}{15} - \frac{53}{15} \right) = \boxed{\frac{153\pi}{5}}$

Math 250

(9) $y = 4 + 2x - x^2$
around $y = 1$



$y = -x^2 + 2x + 4$
vertex $\frac{-2}{2(-1)} = 1$

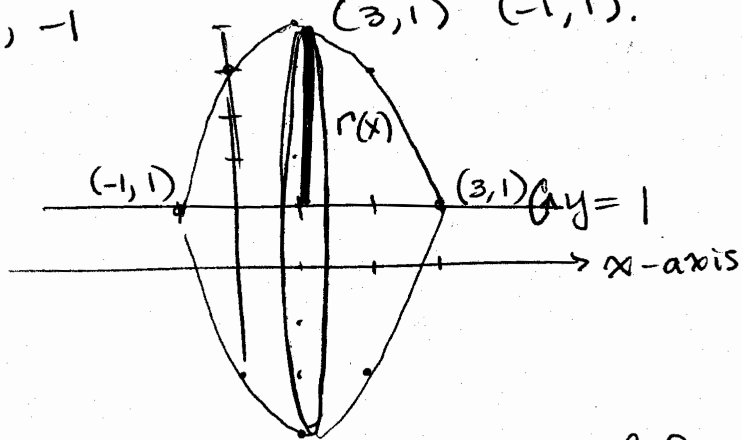
$y(1) = -1^2 + 2(1) + 4$
$= -1 + 2 + 4$
$= 5$

vertex $(1,5)$
downward $a<0$
standard shape

Points of intersection $1 = 4 + 2x - x^2$
$x^2 - 2x - 3 = 0$
$(x-3)(x+1) = 0$
$x = 3, -1$

radius is a vertical bar
$=$ top y − bottom y
$= (4+2x-x^2) - (1)$

$(3,1)$ $(-1,1)$.

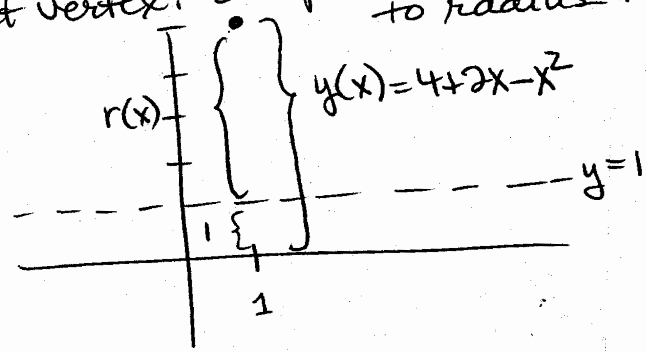


Revolve around $y = 1$
radius is perpendicular to axis of revolution.

radius $\neq$ y-coord
$4 + 2x - x^2$.

$r(x) = y(x) - 1$

at vertex: comparision of function to radius:



$$\int_{-1}^{3} \pi \left(r(x) \right)^2 dx$$

$$= \int_{-1}^{3} \pi \left(y(x) - 1 \right)^2 dx$$ subst top−bottom

$$= \int_{-1}^{3} \pi \left(4 + 2x - x^2 - 1 \right)^2 dx$$ subst for function

Math 250

$$= \pi \int_{-1}^{3} (-x^2+2x+3)^2 \, dx$$ simplify

$$(-x^2+2x+3)(-x^2+2x+3)$$
$$= x^4 - 2x^3 - 3x^2$$
$$- 2x^3 + 4x^2 + 6x$$
$$- 3x^2 + 6x + 9$$
$$= x^4 - 4x^3 - 2x^2 + 12x + 9$$

$$= \pi \int_{-1}^{3} x^4 - 4x^3 - 2x^2 + 12x + 9 \, dx$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{4}{4}x^4 - \frac{2}{3}x^3 + \frac{12}{2}x^2 + 9x \right] \Big|_{-1}^{3}$$ antidiff

$$= \pi \left[\frac{1}{5}x^5 - x^4 - \frac{2}{3}x^3 + 6x^2 + 9x \right] \Big|_{-1}^{3}$$ simplify

$$= \pi \left[\frac{153}{5} - \frac{-53}{15} \right]$$ evaluate

$$= \boxed{\frac{512}{15}\pi} \approx 107.233$$

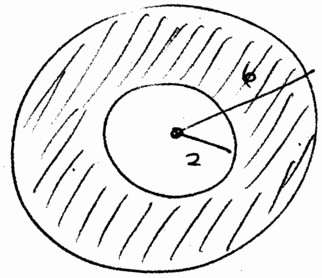
GC says $34.1\overline{3}\ \pi$ ≈ 107.233.

$$\pi \int_{-1}^{3} (-x^2+2x+3)^2 dx$$

Math 250

Background:

Find area of a washer (ring)



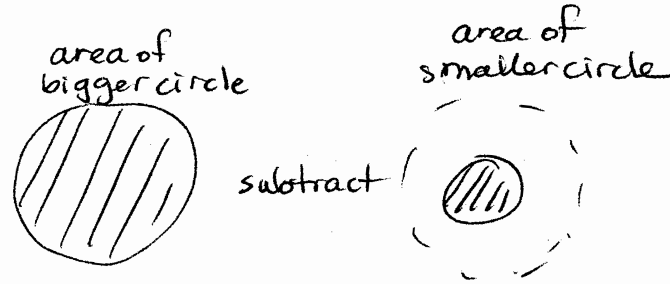
$$A = \pi R^2 - \pi r^2$$

outer area − inner area

$$= \pi(6)^2 - \pi(2)^2$$

$$= 36\pi - 4\pi$$

$$= \boxed{32\pi}$$



Is this the same as

$$\pi(R-r)^2 \ ?$$

$$\pi(R^2 - 2rR + r^2) \qquad \text{No!}$$

middle term is absent in the correct work.

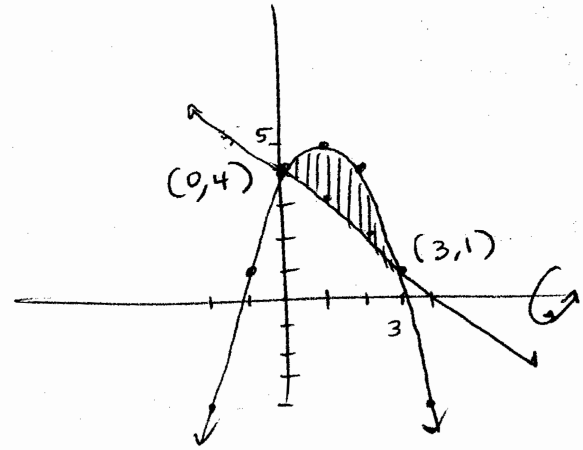
** We will use $\pi R^2 - \pi r^2$ to calculate cross-sections that are rings/washers.

Math 250

(1a) a) $y = 4 + 2x - x^2$

$y = 4 - x$

around x-axis



points of intersection

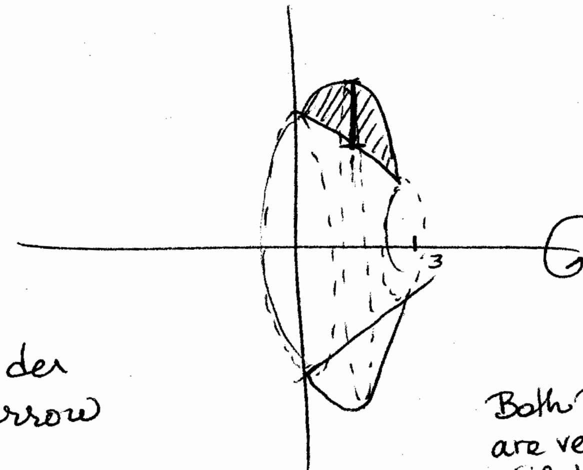
$4 + 2x - x^2 = 4 - x$

$0 = x^2 - 3x$

$0 = x(x-3)$

$x = 0, 3$

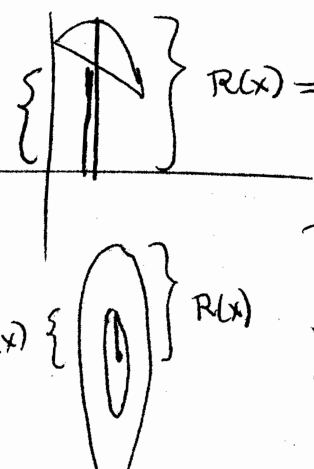
$(0,4) \quad (3,1)$



Revolve → a sideways flying saucer, a hole all the way through the center — wider hole on left, narrow on right

Both R and r are vertical bars with bottom $y=0$.

Both radii are perpendicular to axis of revolution.



$$\int_0^3 \pi R^2 - \pi r^2 \, dx$$ ← area of ring or washer

R(x) must be above (or to right if R(y)) of r(x) (or r(y))

$$= \int_0^3 \pi(4+2x-x^2)^2 - \pi(4-x)^2 \, dx \quad \text{subst}$$

$(4+2x-x^2)^2$
$= 16 + 8x - 4x^2$
$+ 8x + 4x^2 - 2x^3$
$- 4x^2 - 2x^3$
$+ x$

$$= \pi \int_0^3 (x^4 - 4x^3 - 4x^2 + 16x + 16) - (x^2 - 8x + 16) \, dx$$ ← simp

$$= \pi \int_0^3 \begin{matrix} x^4 - 4x^3 - 4x^2 + 16x + 16 \\ -x^2 + 8x - 16 \end{matrix} \, dx$$ multiply

$$= \pi \int_0^3 x^4 - 4x^3 - 5x^2 + 24x + 0 \, dx$$ combine

$$= \pi \left[\frac{1}{5}x^5 - \frac{4}{4}x^4 - \frac{5}{3}x^3 + \frac{24}{2}x^2 \right] \Big|_0^3$$ antidiff

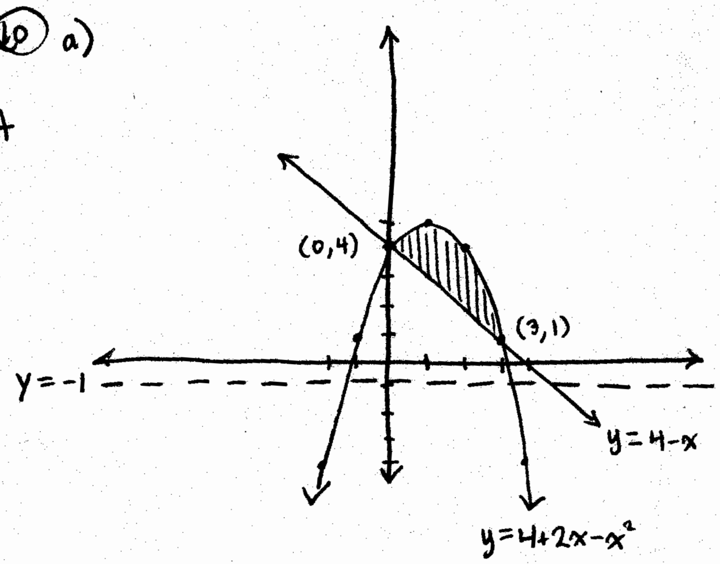
$$= \pi \left[\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right] \Big|_0^3$$ simplify

$$= \boxed{\frac{153\pi}{5}}$$ evaluate $F(b) - \underbrace{F(a)}_{0}$

(10) b) $\left.\begin{array}{l} y = 4 + 2x - x^2 \\ y = 4 - x \end{array}\right\}$ Same as (10) a)

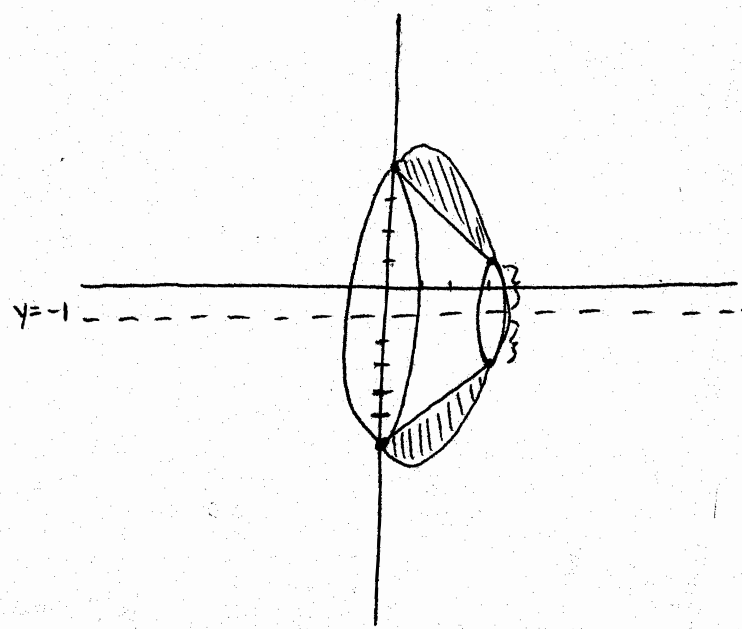
around $y = -1$ } different

Same pts of intersection



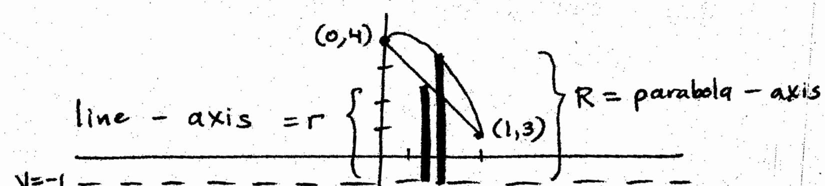
Revolve on $y = -1$

Still a sideways flying saucer



Both radii are perpendicular to the axis of revolution.

Each radius is measured from the graph to the axis of revolution.



r = top - bottom
line - axis

$(4-x) - (-1)$

$= 4 - x + 1$

$= 5 - x$

R = top - bottom
parabola - axis

$4 + 2x - x^2 - (-1)$

$= 4 + 2x - x^2 + 1$

$= 5 + 2x - x^2$

$$\int_{x=0}^{x=3} \pi R^2 - \pi r^2 \, dx$$

$$= \int_0^3 \pi(5+2x-x^2)^2 - \pi(5-x)^2 \, dx$$

$$= \pi \int_0^3 (25+20x-6x^2-4x^3+x^4) - (25-10x+x^2) \, dx$$

$$= \pi \int_0^3 25+20x-6x^2-4x^3+x^4-25+10x-x^2 \, dx$$

algebra $(5+2x-x^2)^2$

$$\begin{array}{rl} = & 25 + 10x - 5x^2 \\ & +10x + 4x^2 - 2x^3 \\ & -5x^2 - 2x^3 + x^4 \\ \hline = & 25 + 20x - 6x^2 - 4x^3 + x^4 \end{array}$$

$$= \pi \int_0^3 x^4 - 4x^3 - 7x^2 + 30x \; dx$$ combine like terms

$$= \pi \left[\frac{1}{5}x^5 - 4 \cdot \frac{1}{4}x^4 - 7 \cdot \frac{1}{3}x^3 + 30 \cdot \frac{1}{2}x^2 \right] \Big|_0^3$$ anti diff

$$= \pi \left[\frac{1}{5}x^5 - x^4 - \frac{7}{3}x^3 + 15x^2 \right] \Big|_0^3$$ simplify

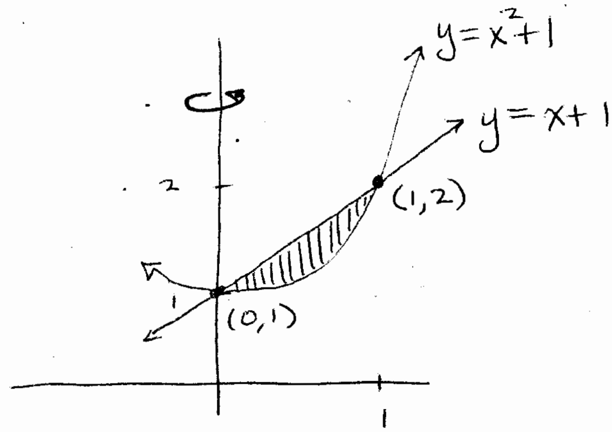
$$= \pi \left[\frac{1}{5}(3)^5 - (3)^4 - \frac{7}{3}(3)^3 + 15(3)^2 \right] - 0$$ evaluate $F(b) - F(a)$

$$= \pi \cdot \frac{198}{5}$$ arithmetic

$$= \boxed{\frac{198\pi}{5}}$$

Math 250

(11) $y = x^2+1$
$y = x+1$
around y-axis.

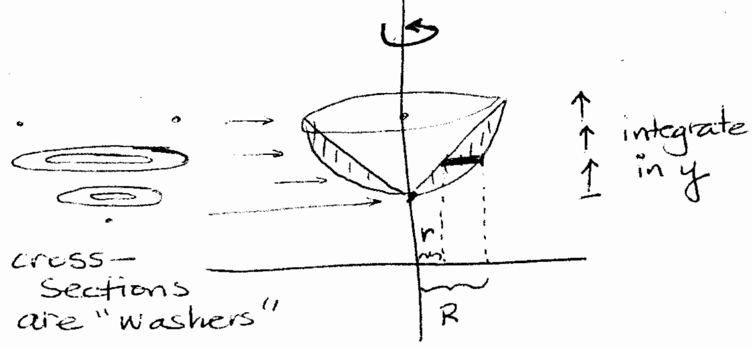


step 1: sketch graph and determine points of intersection

step 2: Washers around y-axis

$$\int_{y=1}^{y=2} \pi R^2 - \pi r^2 \, dy$$

must write the radii as functions of y



R = x-coord of the parabola
r = x-coord of the line

Both R and r are horizontal bars $\Rightarrow$ right x − left x, in terms of y.

Solve each equation for x!

$y = x^2+1$
$y-1 = x^2$
$\pm\sqrt{y-1} = x$
$R = x = \sqrt{y-1}$ right side

$y = x+1$
$y-1 = x$
$r = x = y-1$

$$= \int_{y=1}^{y=2} \pi\left(\sqrt{y-1}\right)^2 - \pi\left(y-1\right)^2 dy$$ subst for R(y) and r(y)

$$= \pi \int_1^2 \left(\sqrt{y-1}\right)^2 - (y-1)^2 \, dy$$ move constant multiple to front of integral

$$= \pi \int_1^2 y-1-(y^2-2y+1)\, dy$$ square

$$= \pi \int_1^2 y-1-y^2+2y-1 \, dy$$ dist neg

Math 250

$$= \pi \int_1^2 -y^2 + 3y - 2 \, dy$$ combine

$$= \pi \left[-\frac{y^3}{3} + \frac{3}{2}y^2 - 2y \right]\Big|_1^2$$ antidiff

$$= \pi \left[\left(-\frac{(2)^3}{3} + \frac{3}{2}(2)^2 - 2(2) \right) - \left(-\frac{1}{3} + \frac{3}{2} - 2 \right) \right]$$ evaluate

$$= \pi \left[-\frac{8}{3} + 6 - 4 + \frac{1}{3} - \frac{3}{2} + 2 \right]$$ simplify

$$= \boxed{\frac{\pi}{6}}$$ arithmetic

Math 250

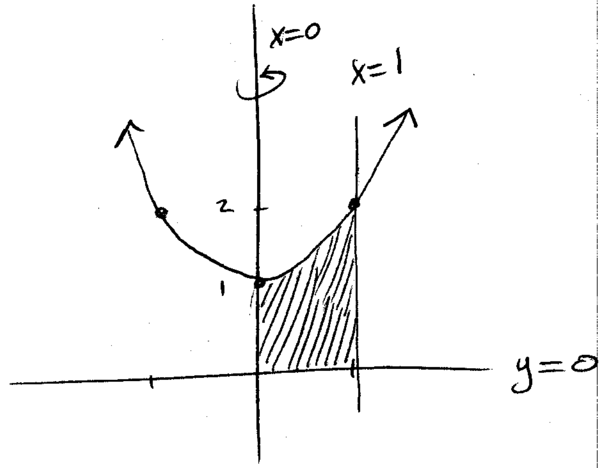
(12) $y = x^2+1$

$y = 0$

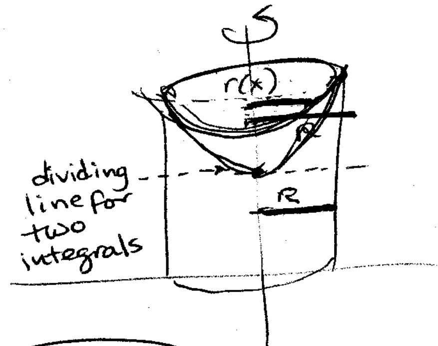
$x = 0$

$x = 1$

around y-axis.



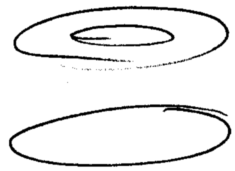
Two different cross-sectional shapes/boundaries when we consider radius perpendicular to axis of revolution.

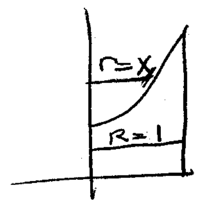


a solid with a rounded indent in top.

above $y=1$, it's a washer

below $y=1$, its a disk





Option 1: Two integrals.

$$\int_{y=0}^{y=1} \pi (R(y))^2 \, dy + \int_{y=1}^{2} \pi (\underbrace{R(y)}_{})^2 - \pi (\underbrace{r(y)}_{})^2 \, dy$$

R=1 doesn't change as y change

The easiest $r=x$ but we must integrate in y — so solve for x:

$y = x^2+1$

$y-1 = x^2$

$x = \pm\sqrt{y-1}$

$x = \sqrt{y-1}$ in QI, (+)

Both R and r are horizontal bars: right x — $\underbrace{\text{left } x}_{\text{y-axis } x=0}$

$$= \underbrace{\int_0^1 \pi (1)^2 \, dy}_{\text{lower cylinder (disk)}} + \underbrace{\int_1^2 \pi (1)^2 - \pi (\sqrt{y-1})^2 \, dy}_{\text{upper washers}}$$

R(y) r(y)

Math 70

$$= \pi \int_0^1 dy + \pi \int_1^2 1 - (y-1)\, dy$$ squares

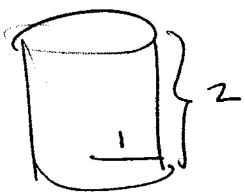
$$= \pi \int_0^1 dy + \pi \int_1^2 2-y\, dy$$ combine

$$= \pi \left[y \Big|_0^1 + \left(2y - \frac{y^2}{2}\right) \Big|_1^2 \right]$$ antidiff

$$= \pi \left\{ \left[1 \right] + \left[(4-2) - \left(2 - \frac{1}{2}\right) \right] \right\}$$ evaluate

$$= \boxed{\frac{3}{2}\pi}$$ arithmetic

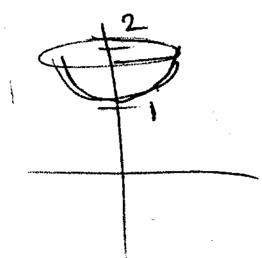
Option 2: Find area of outer cylinder, subtract the space where the indent is.



outer cylinder $= \pi r^2 h$

$$= \pi (1)^2 2$$

$$= 2\pi$$



indent $\displaystyle\int_{y=1}^{y=2} \pi (r(y))^2\, dy$

$$= \pi \int_1^2 (\sqrt{y-1})^2\, dy$$

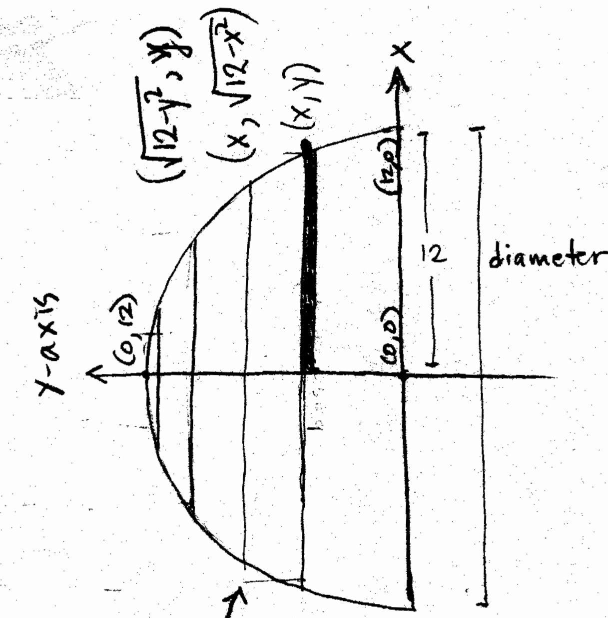
$$= \pi \int_1^2 y-1\, dy$$

$$= \pi \left[\frac{y^2}{2} - y \right] \Big|_1^2 = \pi\left((2-2) - \left(\frac{1}{2} - 1\right)\right) = \frac{\pi}{2}$$

subtract $\Rightarrow 2\pi - \frac{\pi}{2} = \boxed{\frac{3\pi}{2}}$

6.3.11

HW hint

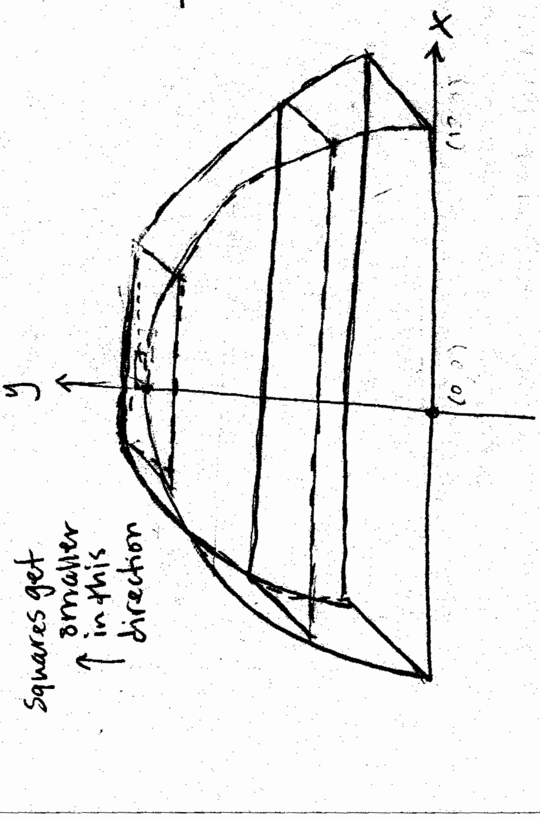


$x^2+y^2=12$

means, $y=\sqrt{12-x^2}$

or $x=\sqrt{12-y^2}$

cross sections
parallel to diameter.
Come out of page toward you.
Each of these lines is the bottom
side of a square



$$\int_{y=0}^{12} (\text{side})^2\,dy = \int_{y=0}^{12} (2\cdot\text{horiz bar})^2\,dy$$